EXPANSIONS IN GENERALIZED APPELL POLYNOMIALS, AND A CLASS OF RELATED LINEAR FUNCTIONAL EQUATIONS*

BY

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1. Introduction. In this paper we are primarily concerned with the solution of the linear differential equation of infinite order

$$\sum_{n=0}^{\infty} P_n(x) y^{(n)}(x) = f(x), \tag{1}$$

where the $P_n(x)$ are polynomials of bounded degree ($\leqq k$), and the relation of this equation to expansions in generalized Appell polynomials. Equation (1) is transformed into an equivalent contour integral equation, using the Laplace transformation; and this integral equation is then shown to lead to a solution which is itself a contour integral, with a kernel which satisfies a linear differential equation of order k. It is also shown that the contour integral equation is equivalent to an expansion question in generalized Appell polynomials.

In §2 we derive some simple properties of these polynomials. In §3 the equivalence between the above-mentioned differential and integral equations is shown, and the relation to the expansion problem developed. The resolving kernel for the general case (k) is introduced in §4, and the particular cases $k=0$, $k=1$, are treated in §§5, 6. The method of the Laplace transformation is then extended, in §§7, 8 respectively, to partial differential equations of infinite order, constant coefficients, and to Laurent differential equations, constant coefficients.‡

2. Generalized Appell polynomials. Let

$$A_i(t) \sim \sum_{n=0}^{\infty} \alpha_{in} t^n \qquad (i = 0, 1, \cdots, k) \tag{2}$$

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‡ The attention of the writer has been called to two memoirs by S. Pincherle, the spirit of which is closely akin to that of the present paper: *Studi sopre alcune operazioni funzionali*, Memorie della Reale Accademia delle Scienze di Bologna, (4), vol. 7 (1886), pp. 393–442; *Sur la résolution de l'équation fonctionnelle* $\sum h_\nu \phi(x+\alpha_\nu) = f(x)$, Acta Mathematica, vol. 48 (1926), pp. 279–304.

be $(k+1)$ formal* power series, with $A_k(t)\not\equiv 0$, and define the *generalized Appell polynomials* $\{G_n(x)\}$, *of order* k, *by the (formal) expansion*†

$$(3)\qquad e^{tx}\{A_0(t)+xA_1(t)+\cdots+x^kA_k(t)\}\sim\sum_{n=0}^{\infty}G_n(x)t^n.$$

If $A_k(0)\neq 0$, $G_n(x)$ is a polynomial of degree $n+k$. If we multiply (5) through by e^{-tx}, and differentiate $(k+1)$ times with respect to x, we find that

$$0\sim e^{-tx}\sum_{n=0}^{\infty}t^n\left\{G_n^{(k+1)}(x)-\binom{k+1}{1}G_{n-1}^{(k)}(x)+\cdots+(-1)^{k+1}G_{n-k-1}(x)\right\},$$

so that

LEMMA 1. $G_n(x)$ *satisfies the mixed relation*‡

$$(4)\qquad \begin{aligned}G_n^{(k+1)}(x)&-\binom{k+1}{1}G_{n-1}^{(k)}(x)+\binom{k+1}{2}G_{n-2}^{(k-1)}(x)\\&-\cdots+(-1)^{k+1}G_{n-k-1}(x)=0\qquad(n=0,1,\cdots).\end{aligned}$$

On expanding the left hand member of (3) in a power series in t we obtain an explicit form for $G_n(x)$:

$$(5)\qquad G_n(x)=\sum_{i=0}^{k}x_i\left(a_{i0}\frac{x^n}{n!}+a_{i1}\frac{x^{n-1}}{(n-1)!}+\cdots+a_{in}\right).$$

Not only does (3) imply (4), but also conversely:

LEMMA 2. *Let* $\{G_n(x)\}$, $n=0,1,\cdots$, *be a set of functions satisfying* (4) *for* $n=0,1,\cdots$. *Then there exist constants* $a_{in}(n=0,1,\cdots,\infty;i=0,1,\cdots,k)$ *such that on setting (formally)* $A_i(t)\sim\sum_0^\infty a_{in}t^n$, *the relation* (3) *is formally satisfied.*

In virtue of (4) we have

$$0\sim e^{-tx}\sum_{n=0}^{\infty}\left\{G_n^{(k+1)}(x)-\binom{k+1}{1}G_{n-1}^{(k)}(x)+\cdots+(-1)^{k+1}G_{n-k-1}(x)\right\}.$$

But the right hand member of this identity is the $(k+1)$st derivative with

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* That is, the radius of convergence of $A_i(t)$ may be zero.

† The original Appell polynomials $(k=0)$ were discussed by Appell, *Sur une classe de polynomes*, Annales Scientifiques de l'Ecole Normal Supérieure, (2), vol. 9 (1880), pp. 119–144. He has found for them some interesting properties.

‡ G's with negative subscripts are defined to be identically zero.

respect to x of $e^{-tx}\sum_0^\infty t^n G_n(x)$, so that this last expression is a polynomial in x of degree not exceeding k, with coefficients which are functions of t:*

$$e^{-tx}\sum_0^\infty t^n G_n(x) \sim A_0(t) + xA_1(t) + \cdots + x^k A_k(t).$$

It follows that the functions $G_n(x)$ are *polynomials*. To determine the functions $A_i(t)$, we differentiate the members of this last identity k times with respect to x, setting $x=0$ after each differentiation:

$$A_0(t) \sim \sum_0^\infty G_n(0)t^n,$$

$$A_1(t) \sim \frac{1}{1!}\sum_0^\infty \left\{G_n'(0) - G_{n-1}(0)\right\}t^n,$$

$$\cdots\cdots\cdots\cdots$$

$$A_k(t) \sim \frac{1}{k!}\sum_0^\infty \left\{G_n^{(k)}(0) - \binom{k}{1}G_{n-1}^{(k-1)}(0) + \binom{k}{2}G_{n-2}^{(k-2)}(0) - \cdots + (-1)^k\binom{k}{k}G_{n-k}(0)\right\}t^n.$$

This establishes the lemma.

COROLLARY. *A necessary and sufficient condition that a set of functions* $\{G_n(x)\}$, $n=0, 1, \cdots$, *be a set of generalized Appell polynomials of order* $\leqq k$, *is that* (4) *be satisfied*, $n=0, 1, 2, \cdots$

In establishing the preceding results the question of convergence of the formal power series represented by $A_i(t)$ did not enter. However, in considering G_n-expansions of analytic functions, it is necessary to lay down restrictions. We make the following assumptions:

(A) The functions $A_i(t)$, $i=0, 1, \cdots, k$, are analytic, $|t|<R$.

(B) $A_k(t)$ has at most a finite number† of zeros in $|t|<R$.

(C) The analytic functions $f(x)$ whose expansions we consider are those of *exponential value* (*exp. val.*) less than R; i.e., those for which $\limsup_{n\to\infty} |f^{(n)}(0)|^{1/n}<R$.

LEMMA 3. *If* $\{\lambda_n\}$ *is a sequence of numbers such that* $\limsup |\lambda_n|^{1/n}=1/\rho<R$, *then* $\sum_0^\infty \lambda_n G_n(x)$ *converges uniformly in every bounded region, and defines a function of exponential value* $<R$.

* It is possible that $A_k(t)\equiv 0$, $A_{k-1}(t)\equiv 0, \cdots, A_{k-r}(t)\equiv 0$, $0\leqq r\leqq k$.

† If $A_k(t)$ has infinitely many zeros in $|t|<R$, then we choose R' very close to, *and less than*, R, and consider the region $|t|<R'$. There will be only a finite number of zeros in $|t|<R'$.

Let r be any number $<R$, and let C be the circle $|t|=r$. Then in any bounded region $|x|\leqq X$,

$$|G_n(x)| = \left| \frac{1}{2\pi i}\int_C e^{tx}\{A_0(t) + \cdots + x^k A_k(t)\}\frac{dt}{t^{n+1}}\right|$$
$$\leqq e^{rX}\cdot N(1 + X + \cdots + X^k)/r^n = M/r^n,$$

where N is the maximum of $|A_0(t)|, \cdots, |A_k(t)|$ on C. Choose r to satisfy $r>1/\rho$. Then in $|x|\leqq X$, $\sum_0^\infty |\lambda_n G_n(x)|\leqq M\sum_0^\infty |\lambda_n|/r^n$, which converges. The series represents, then, an entire function. Denote it by $g(x)$. Now $|a_{in}|\leqq M/(R-\epsilon)^n$, $i=0, \cdots, k$, so that

$$A_0(t) + \cdots + x^k A_k(t) \ll M(1 + x + \cdots + x^k)\sum_0^\infty t^n(R-\epsilon)^{-n},$$

and

$$e^{tx}\{A_0(t) + \cdots + x^k A_k(t)\} \ll M(1 + x + \cdots$$
$$+ x^k)\sum_0^\infty t^n[x^n/n! + \cdots + (R-\epsilon)^{-n}].$$

Hence

$$G_n(x) \ll M(1 + x + \cdots + x^k)[x^n/n! + \cdots + (R-\epsilon)^{-n}].$$

Also, $|\lambda_n|\leqq N(\rho-\epsilon)^{-n}$; therefore

$$\sum_{n=0}^\infty \lambda_n G_n(x) \ll MN(1+x+\cdots+x^k)\sum_{n=0}^\infty (x^n/n!)(\rho-\epsilon)^{-n}\left\{\sum_{s=0}^\infty (R-\epsilon)^{-s}(\rho-\epsilon)^{-s}\right\}.$$

The series in the brace converges, since $R\rho>1$ and ϵ can be taken arbitrarily small; let its value be K. Then

$$g(x) = \sum_0^\infty \lambda_n G_n(x) \ll MNK(1 + x + \cdots + x^k)e^{x/(\rho-\epsilon)}.$$

Now exp. val. of $e^{x/(\rho-\epsilon)}$ is $1/(\rho-\epsilon)$, $<R$ for ϵ small enough, and multiplying through by $(1+x+\cdots+x^k)$ does not affect the exponential value. Hence $g(x)$ is of exp. val. $<R$, as was to be proved.

3. **A differential equation of infinite order and a contour integral equation.** We return to the differential equation (1). Let us denote by $A[y(x)]$ the differential operator

$$A[y] \equiv a_0 y(x) + \cdots + a_n y^{(n)}(x) + \cdots,$$

where $A(t)\sim\sum_0^\infty a_n t^n$. Then equation (1) can be expressed, by a suitable rearrangement of terms, as follows:

$$A_0[y] + xA_1[y] + \cdots + x^k A_k[y] = f(x). \tag{6}$$

We assume in what follows that conditions (A), (B), (C) *of the preceding section are fulfilled.*

Such equations have been considered by Perron,* Hilb, von Koch and others. We shall treat them from a new point of view. Let us look for solutions $y(x)$ which are of exp. val. $<R$. If $y(x)=\sum_0^\infty \lambda_n x^n/n!$ is of exp. val. $\mu<R$, and if $Y(x)=\sum_n^\infty \lambda_n x^n$, then $Y(x)$ has a radius of convergence $1/\mu$; and if C is a contour surrounding the origin and lying in $|t|<1/\mu$, then

$$y(x) = \frac{1}{2\pi i}\int_C \frac{Y(t)}{t} e^{x/t}dt. \tag{7}$$

Now it is readily established that if $A(t)$ is analytic in $|t|<R$, and $y(x)$ is of exp. val. $\mu<R$, then†

$$A[y] = \frac{1}{2\pi i}\int_C \frac{Y(t)}{t} A\left(\frac{1}{t}\right) e^{x/t}dt, \quad 1/R < |t| < 1/\mu, \tag{8}$$

and $A[y]$ is of exp. val. $\leqq \mu$. That is, the operator A can be taken under the integral sign. It follows that (6) *is equivalent to the contour integral equation*

$$f(x) = \frac{1}{2\pi i}\int_C \frac{Y(t)}{t}\left\{A_0\left(\frac{1}{t}\right) + xA_1\left(\frac{1}{t}\right) + \cdots + x^k A_k\left(\frac{1}{t}\right)\right\} e^{x/t}dt, \tag{9}$$

the sense of the equivalence being as follows:

Theorem 1. *If $y(x)=\sum_0^\infty \lambda_n x^n/n!$, of exp. val. $\mu<R$, is a solution of* (6) *then $Y(x)=\sum_0^\infty \lambda_n x^n$ is a solution of* (9), *C being a contour surrounding the origin and lying in $1/R<|t|<1/\mu$. Conversely, if $Y(x)$, with radius of convergence $1/\mu$, where $\mu<R$, is a solution of* (9), *then $y(x)$ is of exp. val. μ, and is a solution of* (6).‡

Let us assume in (9) the expansion

$$Y(t) = \sum_0^\infty \lambda_n t^n. \tag{10}$$

The term in $1/t$ in the integrand is the term independent of t in the Laurent expansion of $Y(t)\sum_0^\infty G_n(x)/t^n$, so that we have

$$f(x) = \sum_0^\infty \lambda_n G_n(x). \tag{11}$$

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* For a list of references see Sheffer, *Linear differential equations of infinite order, with polynomial coefficients of degree one,* to appear in the Annals of Mathematics.

† $A[e^{\lambda x}]=A(\lambda)e^{\lambda x}$ for all $|\lambda|<R$.

‡ The proof is immediate.

There is, then, a very close relation between equations (6) and (9) on the one hand and G_n-expansions on the other. In precise form:

THEOREM 2. *Let* $f(x)$ *be of exp. val.* $<R$. *If* $y(x)=\sum_0^\infty \lambda_n x^n/n!$, *of exp. val.* $<R$, *is a solution of* (6), *then* $f(x)$ *possesses the expansion* (11), *which converges uniformly in every bounded region*; *and* $\limsup |\lambda_n|^{1/n}<R$. *Conversely, if* $\limsup |\lambda_n|^{1/n}<R$, *and* $f(x)$ *is defined by* (11), *then* $f(x)$ *and* $y(x)=\sum_0^\infty \lambda_n x^n/n!$ *are of exp. val.* $<R$, *and* $y(x)$ *is a solution of* (6).

The theorem follows at once on using (9) and Lemma 3.

DEFINITION. An expansion for $f(x)$: $f(x)=\sum_0^\infty \lambda_n G_n(x)$ will be termed a *proper expansion* (with respect to the given set $\{G_n(x)\}$) if $\limsup |\lambda_n|^{1/n}<R$.*

Theorem 2 relates to proper expansions, and may be (partially) restated as follows:

COROLLARY. *Let* $f(x)$ *be of exp. val.* $<R$. *Every proper expansion of* $f(x)$ *is equivalent to a solution* (*of exp. val.* $<R$) *of equation* (6).

A remarkable property of G_n-expansions is that they may permit of zero expansions; i.e., the function $f(x)\equiv 0$ may have one or more linearly independent expansions in which not all the coefficients vanish. This property is a consequence of the fact that the homogeneous equation corresponding to (6) may have solutions other than $y(x)\equiv 0$:

THEOREM 3.† *Every proper expansion of the function zero, in which not all the coefficients vanish, is equivalent to a solution* $y(x)\not\equiv 0$, *of exp. val.* $<R$, *of the homogeneous equation*

$$A_0[y] + xA_1[y] + \cdots + x^k A_k[y] = 0.$$

For equation (6) Perron‡ has obtained the following theorem:

Let m be the number of zeros (multiple zeros counted multiply) of $A_k(t)$ in $|t|<R$, and let s be the number of linearly independent solutions $D(t)$, which are analytic everywhere in $|t|<R$, of the equation

$$A_0(t)D(t) + A_1(t)D'(t) + \cdots + A_k(t)D^{(k)}(t) = 0. \tag{12}$$

Then the number of linearly independent solutions $y(x)$, of exp. val. $<R$, of the equation

* It follows by Lemma 3 that if $f(x)$ has a proper expansion, then $f(x)$ is of exp. val. $<R$.

† The proof follows that of Theorem 2, the expansion $0=\sum_0^\infty \lambda_n G_n(x)$ corresponding to the solution $y(x)=\sum_0^\infty \lambda_n x^n/n!$

‡ Perron, *Lineare Differentialgleichungen unendlich hoher Ordnung mit ganzen rationalen Koeffizienten*, Mathematische Annalen, vol. 84 (1921), pp. 31–42.

$$A_0[y] + xA_1[y] + \cdots + x^k A_k[y] = 0 \tag{13}$$

is precisely $m-k+s$; and the non-homogeneous equation (6) will possess a solution of exp. val. $<R$ for *all* functions $f(x)$ of exp. val. $<R$ if and only if (12) has no solution analytic everywhere in $|t|<R$.*

Applying this existence theorem and Theorem 2 (Corollary) and Theorem 3, we have

THEOREM 4. *A necessary and sufficient condition that a proper expansion exist for all functions $f(x)$ of exp. val. $<R$ is that equation* (12) *have no solution $D(t)\not\equiv 0$ which is everywhere analytic in $|t|<R$. The number of linearly independent*† *proper expansions of the function zero is precisely*‡ $m-k+s$.

4. **The resolving kernel.**§ By Theorem 4, if (12) has a solution $D(t)$ analytic in $|t|<R$, not every function $f(x)$ of exp. val. $<R$ has a proper expansion. Let us then consider the contrary case: where every solution $D(t)$ has some singularity in $|t|<R$, so that solutions of (6) always exist.

We wish to invert the integral equation (9). Now (9) is in the form

$$f(x) = \frac{1}{2\pi i}\int_C \left\{ Y(t)\Theta\left(\frac{1}{t}\,;\,x\right)\right\}\frac{dt}{t}\,.$$

If then we interchange the rôles of f and Y (which is to introduce F and y), we are led to consider as a solution of the original equation (6) an expression of the form

$$y(x) = \frac{1}{2\pi i}\int_C \frac{F(t)}{t}\Lambda\left(\frac{1}{t}\,;\,x\right)dt, \tag{14}$$

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* That not *all* functions $f(x)$ yield a solution of (6) in the case that (12) has a solution analytic everywhere in $|t|<R$, may be seen as follows:

If $A(t)$, $B(t)$ are analytic in $|t|<R$, and $y(x)$ is of exp. val. $<R$, it is permissible to operate with A on $x^iB[y]$:

$$A[x^iB[y]] \equiv x^iAB[y] + \binom{i}{1}x^{i-1}A'B[y] + \cdots + \binom{i}{i}A^{(i)}B[y].$$

Here $A^{(l)}$ stands for the operator obtained by differentiating $A(t)=\sum_0^\infty a_nt^n$ l times, and $A^{(l)}B$ is the operator defined by the product $A^{(l)}(t)B(t)$, the multiplication of these power series being as usual.

Let $D(t)\not\equiv 0$ satisfy (12) and be analytic in $|t|<R$. Then on operating on (6) with D, as we may, we find that the term on the left which has no factor x, drops out in virtue of (12), so that the left hand member has a factor x. The right hand member is $D[f(x)]$, and this will not in general vanish at the origin. Consequently there will not always be a solution $y(x)$ of exp. val. $<R$.

† It is clear that linearly independent solutions yield linearly independent expansions.

‡ If $m-k+s$ is negative, zero will be understood.

§ We owe to Professor Tamarkin suggestions which have improved the presentation of this section, particularly in stressing the fact that the resolving kernel is not always unique.

where C is a contour surrounding $t=0$ and lying in $1/R<|t|<1/\lambda$, $f(x)$ being of exp. val. λ. The function $\Lambda(t;x)$ is to be determined.

On substituting (14) into (6) we arrive at the equation

$$f(x) = \frac{1}{2\pi i}\int_C \frac{F(t)}{t}\{A_0[\Lambda] + xA_1[\Lambda] + \cdots + x^kA_k[\Lambda]\}dt,$$

and this will certainly be true (see (7)) if the brace in the integrand reduces to $e^{x/t}$. The function $\Lambda(t;x)$ is then to satisfy (with respect to x) the differential equation of infinite order

$$(15) \qquad A_0[\Lambda(t;x)] + xA_1[\Lambda(t;x)] + \cdots + x^kA_k[\Lambda(t;x)] = e^{tx}.$$

DEFINITION. A function $\Lambda(t;x)$ which satisfies (15), is analytic in t for all t in *T, and is (in x) of exp. val. $<R$ for all t in T, will be termed a *resolving kernel*, since it provides in (14) a solution of (6).

Consider the homogeneous equation

$$(16) \qquad A_0[z(x)] + xA_1[z(x)] + \cdots + x^kA_k[z(x)] = 0.$$

If $m-k+s>0$, this will be (by the Perron existence theorem) the number of linearly independent solutions $z_i(x)$ of (16), of exp. val. $<R$. It follows that there will not then be a unique resolving kernel:

LEMMA 4. *If there exists a resolving kernel, and $\Lambda(t;x)$ is such a one, then so is*

$$\text{(a)} \quad \Lambda(t;x)+\textstyle\sum_{i=1}^{r} \Upsilon_i(t)z_i(x),$$

if $r=m-k+s>0$; and this is the most general resolving kernel. Here the functions $\Upsilon_i(t)$ are subject only to the condition that they be analytic for t in T.

If $m-k+s\leqq 0$, (16) has no solution of exp. val. $<R$ other than $z(x)\equiv 0$. Hence

LEMMA 4′. *If $m-k+s\leqq 0$, a resolving kernel, if it exists, is unique.*

Let us denote by A_x the operator on the left of (15), so that $A_x[\Lambda(t;x)] = e^{tx}$; and let B_t be the operator

$$(17) \qquad B_t[w(t)] \equiv A_0(t)w + A_1(t)w' + \cdots + A_k(t)w^{(k)}.$$

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* T is the region $|t|<R$ with the zeros of $A_k(t)$ deleted. It is not strictly necessary to demand the analyticity and exp. val. conditions throughout T: it would suffice to consider only those values of t for which $1/t$ lies on C. But since the contour C may vary in T as we choose different functions $f(x)$ it is simpler to use the region T.

Then

$$A_x[e^{tx}] = B_t[e^{tx}] = e^{tx}\sum_{i=0}^{k} x^i A_i(t),$$

and*

$$A_x[B_t[\Lambda]] = B_t[A_x[\Lambda]] = B_t[e^{tx}] = A_x[e^{tx}], \tag{18}$$

so that

$$B_t[\Lambda(t; x)] = e^{tx} + \phi(t; x), \tag{19}$$

where $\phi(t; x)$ is a solution of $A_x[\phi]=0$. If Λ is a resolving kernel, then it is analytic in t (in the region T) and is (in x) of exp. val. $<R$ (t in T). The same is then true of $B_t[\Lambda]$, and hence of ϕ. That is,†

$$\phi(t; x) = \sum_{i=1}^{r} \Upsilon_i(t) z_i(x), \tag{b}$$

$\Upsilon_i(t)$ analytic in T.

Lemma 5. *There is a solution $\Lambda_1(t; x)$ of the linear differential equation of order k*

$$B_t[\Lambda_1] \equiv A_0(t)\Lambda_1 + A_1(t)\frac{\partial \Lambda_1}{\partial t} + \cdots + A_k(t)\frac{\partial^k \Lambda_1}{\partial t^k} = e^{tx} + \phi(t; x), \tag{20}$$

which is analytic in t (in the region T) and which is (in x) of exp. val. $<R$ for t in T. Here ϕ is given by (b).†

The lemma follows from the properties of linear differential equations of finite order.

Consider such a function $\Lambda_1(t; x)$. We have

$$B_t[A_x[\Lambda_1]] = A_x[B_t[\Lambda_1]] = A_x[e^{tx}] = B_t[e^{tx}],$$

so that

$$A_x[\Lambda_1(t; x)] = e^{tx} + \psi(t; x), \tag{21}$$

where ψ is a solution of $B_t[\psi]=0$. The left hand member of (21) is of exp. val. $<R$ (see (8)). This must then also be true of $\psi(t; x)$.

Let $c_1(t), \cdots, c_k(t)$ be a set of linearly independent solutions of $B_t[c(t)] =0$. ($c_i(t)$ is clearly analytic in T.) Then we can write

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* That $A_x B_t = B_t A_x$ is readily seen.

† If $A_x[z]=0$ has no solution (of exp. val. $<R$) other than $z(x)\equiv 0$, the sum in (b) drops out: $\sum_{i=1}^{r}\Upsilon_i(t)z_i(x)\equiv 0$.

(c) $$\psi(t;x) = \sum_{i=1}^{k} u_i(x)c_i(t),$$

where the $u_i(x)$ are functions of exp. val. $<R$.*

We come now to

THEOREM 5. *Let* (12) *have no solution* $D(t)\not\equiv 0$ *which is analytic everywhere in* $|t|<R$. *Then every function* $\Lambda(t;x)$ *defined by*

(22) $$\Lambda(t\,;x) = \Lambda_1(t\,;x) + \chi(t\,;x),$$

where χ *is an arbitrary solution of*†

(23) $$A_x[\chi(t\,;x)] = -\psi(t\,;x),$$

ψ *being given by* (c), *is a resolving kernel.*

Our hypothesis on equation (12) ensures us (by the Perron existence theorem) that a solution χ exists of the desired character.‡ Equation (22) then defines Λ to be analytic in t (in the region T) and of exp. val. $<R$ for all t in T. We need then only show that Λ satisfies (15). Now $A_x[\Lambda]=A_x[\Lambda_1]-\psi$, and this is, by (21), precisely e^{tx}.

Let us return to equation (15). We see (Lemmas 4 and 4′) that the resolving kernel just obtained is unique unless $m-k+s>0$, and in the case of non-uniqueness Λ may be augmented by $\sum_{i=1}^{r}\rho_i(t)z_i(x)$, the $\rho_i(t)$ being arbitrary functions analytic in T. This result is also reflected in

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* For ψ certainly has the form (c) where the $u_i(x)$ are *some* functions of x. It remains to establish their exponential value. Now there exist k values of t in T: $t=t_1,\cdots,t_k$, such that the determinant Δ of the c_i's at these k points is not zero. For suppose the contrary, and let $t_1,\cdots,t_{k-1}$ be held fast, while t_k varies in T. Then Δ, which is a linear combination of $c_1(t_k),\cdots,c_k(t_k)$, must vanish identically (in t_k). Hence the coefficients of $c_1(t_k),\cdots,c_k(t_k)$ must vanish. This is true for all $t_1,\cdots,t_{k-1}$ in T, and as the coefficients in question are $(k-1)$-order minors of Δ, we have reduced the question to a $(k-1)$-order determinant. If now we permit t_{k-1} to vary in T, holding $t_1,\cdots,t_{k-2}$ fast, we reduce still further. Finally we arrive at a first-order determinant: $c_1(t_1)$, which must vanish identically in t_1. But this is manifestly a contradiction.

If we regard equations (c) (for $t=t_1,\cdots,t_k$) as a set of linear equations in $u_1(x),\cdots,u_k(x)$, we find (since $\Delta\neq 0$) that $u_i(x)=$linear combination of $\psi(t_1;x),\cdots,\psi(t_k;x)$. Since $\psi(t;x)$ is of exp. val. $<R$ for all t in T, it follows then that $u_i(x)$ is of exp. val. $<R$.

† χ is also required to be analytic in t (in the region T) and (in x) of exp. val. $<R$ for all t in T.

‡ For the functions $u_i(x)$ of (c) are of exp. val. $<R$, so that functions $w_i(x)$, likewise of exp. val. $<R$, exist such that $A_x[w_i(x)]=u_i(x)$. Then the most general χ is given by

$$\chi(t;x) = -\sum_{i=1}^{k} c_i(t)w_i(x) + \sum_{i=1}^{r}\sigma_i(t)z_i(x),$$

where the $\sigma_i(t)$ are arbitrary functions which are analytic in T.

COROLLARY 1. *Any function* $\Lambda(t; x)$ *defined by* (22) *satisfies the linear differential equation of order k*

$$(24)\quad B_t[\Lambda] \equiv A_0(t)\Lambda + A_1(t)\frac{\partial\Lambda}{\partial t} + \cdots + A_k(t)\frac{\partial^k\Lambda}{\partial t^k} = e^{tx} + \sum_{i=1}^{r}\mu_i(t)z_i(x),$$

where the $\mu_i(t)$* *are suitably chosen functions analytic in* T, *and the* $z_i(x)$ *are a complete set*† *of linearly independent solutions of* $A_x[z(x)]=0$.

Given Λ defined by (22). Now $A_x[B_t[\chi]]=B_t[A_x[\chi]]=-B_t[\psi]=0$. Hence $B_t[\chi]=\sum_{i=1}^{r}\delta_i(t)z_i(x)$, $\delta_i(t)$ analytic in T. Also,

$$\phi(t; x) = \sum_{i=1}^{r} \Upsilon_i(t)z_i(x),$$

$\Upsilon_i(t)$ analytic in T. Formula (24) then follows from the relation

$$B_t[\Lambda] = B_t[\Lambda_1] + B_t[\chi] = e^{tx} + \{\phi(t; x) + B_t[\chi]\}.$$

COROLLARY 2. *There exists a resolving kernel* $\Lambda(t; x)$ *which satisfies*

$$(24')\qquad A_0(t)\Lambda + A_1(t)\frac{\partial\Lambda}{\partial t} + \cdots + A_k(t)\frac{\partial^k\Lambda}{\partial t^k} = e^{tx}.$$

The lemma asserts that a possible choice in (24) is $\mu_i(t)\equiv 0$. Two cases arise: (i) $m-k+s\leqq 0$. Then in (24) the sum $\sum_{i=1}^{r}\mu_i(t)z_i(x)$ drops out, and we have precisely (24′).

(ii) $m-k+s>0$. Then $\Lambda(t; x)$ is not unique. Let $\Lambda_2(t; x)$ satisfy (24), where all the $\mu_i(t)$ are zero. Then the general Λ is (by Lemma 4) $\Lambda=\Lambda_2+\sum_{i=1}^{r}\rho_i(t)z_i(x)$, $\rho_i(t)$ arbitrary (but analytic in T); and

$$B_t\left[\Lambda_2 + \sum_{i=1}^{r}\rho_i(t)z_i(x)\right] = e^{tx} + \sum_{i=1}^{r}z_i(x)\{\mu_i(t) + B_t[\rho_i(t)]\}.$$

On choosing $\rho_i(t)$ to satisfy $B_t[\rho_i(t)]=-\mu_i(t)$, the sum on the right drops out, and we have (24′).

That $\Lambda(t; x)$ can be augmented by the sum $\sum_{i=1}^{r}\rho_i(t)z_i(x)$ (when $m-k+s>0$) should imply (from (24)) that $B_t[\sum_{i=1}^{r}\rho_i(t)z_i(x)]$ is again of the form $\sum_{i=1}^{r}\beta_i(t)z_i(x)$, $\beta_i(t)$ analytic in T; and this is at once verified.

An immediate consequence of the definition of resolving kernel is

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* The $\mu_i(t)$ will vary as we choose different Λ's.

† We recall that if $m-k+s\leqq 0$, the sum $\sum_{i=1}^{r}\mu_i(t)z_i(x)$ must drop out.

THEOREM 6.* *Let the same assumption on $D(t)$ be made as in Theorem 5. Then every resolving kernel $\Lambda(t; x)$ makes (14) a solution of the original equation (6).*

Returning to G_n-expansions we have

THEOREM 7. *Let $f(x)$ be of exp. val. $\lambda < R$ and let $\Lambda(t; x)$ be the resolving kernel given by Theorem 5.† Then $f(x)$ possesses the proper expansion*

$$f(x) = \sum_0^\infty \lambda_n G_n(x) \tag{25}$$

with

$$\lambda_n = \frac{1}{2\pi i}\int_C \frac{F(t)}{t}\Omega_n\left(\frac{1}{t}\right)dt, \tag{26}$$

where

$$\Lambda(t\,;\,x) = \sum_{n=0}^\infty \frac{\Omega_n(t)x^n}{n!}. \tag{27}$$

Defining $\Omega_n(t)$ by (27) and λ_n by (26), we have, by (14), $y(x) = \sum_0^\infty \lambda_n x^n/n!$. The theorem now follows by use of Theorems 2 and 6.

The investigation of the resolving kernel $\Lambda(t; x)$, when it exists, is better effected through equation (24′) than through (15), since the theory of linear differential equations of finite order is well known. Yet even when dealing with (24′) it is difficult to obtain for the solutions a form which is convenient to handle. We shall content ourselves with a study of the cases $k=0$, $k=1$.

5. **The case $k=0$.** Here (15), (24′) reduce to

$$A_0[\Lambda] = e^{tx}, \tag{15a}$$ $$A_0(t)\Lambda(t; x) = e^{tx}, \tag{24'a}$$

and it is seen that

$$\Lambda(t\,;\,x) = e^{tx}A_0(t) = \sum_{n=0}^\infty \frac{t^n}{A_0(t)}\cdot\frac{x^n}{n!} \tag{27a}$$

---

* It is assumed that $f(x)$ is of exp. val. $\lambda < R$, and that the contour C of (14) surrounds the origin and lies in $1/R < |t| < 1/\lambda$. We naturally choose C so as not to pass through any point $1/\xi$, where ξ is a zero of $A_k(t)$, for at such a point $\Lambda(1/t; x)$ may cease to be analytic. It may be remarked that the formal step of taking the operators $A_0, xA_1, \cdots, x^kA_k$ under the integral sign in passing from (14) to (15) is easily justified.

† The same assumption on $D(t)$ is made as in Theorem 5.

satisfies both these equations. (27a) is then a resolving kernel* for equation (6a):

$$A_0[y] = f(x). \tag{6a}$$

On applying Theorems 6 and 7 we have

THEOREM 8. *If $f(x)$ is of exp. val. $\lambda < R$, then* (6a) *has a solution $y(x)$, of exp. val. $< R$, given by*

$$y(x) = \frac{1}{2\pi i}\int_C \frac{F(t)}{t}\frac{e^{x/t}dt}{A_0(1/t)}, \tag{14a}$$

C being a contour surrounding the origin and lying in $1/R < |t| < 1/\lambda$; and $f(x)$ possesses the proper expansion

$$f(x) = \sum_0^\infty \lambda_n G_n(x) \tag{25a}$$

where†

$$\lambda_n = \frac{1}{2\pi i}\int_C \frac{F(t)}{t^{n+1}}\frac{dt}{A_0(1/t)}. \tag{26a}$$

It is possible to obtain solutions of the homogeneous equation $A_0[y]=0$ (in the case that $A_0(t)$ has at least one zero in $|t|<R$), as contour integrals. If ξ is a zero of $A_0(t)$ (in $|t|<R$), and p is the order, then there are p linearly independent solutions corresponding to this zero: $e^{\xi x}, xe^{\xi x}, \cdots, x^{p-1}e^{\xi x}$; and these give rise to the following p proper expansions of the function zero:‡

$$0 = \sum_0^\infty \lambda_n G_u(x),$$

where $\lambda_n = \xi^n, n\xi^{n-1}, n(n-1)\xi^{n-2}, \cdots, n(n-1)\cdots(n-p+2)\xi^{n-p+1}$.

6. Case $k=1$. For $k=1$ equations (15), (24′) become (15b), $A_0[\Lambda] + xA_1[\Lambda] = e^{tx}$, (24′b), $A_0(t)\Lambda + A_1(t)\partial\Lambda/\partial t = e^{tx}$.

A solution of (24′b) is furnished by

$$y(t;x) = e^{-\int(A_0/A_1)dt}\int\frac{e^{tx+\int(A_0/A_1)dt}}{A_1}dt,$$

and we have

$$A_0[y] + xA_1[y] = e^{-\int(A_0/A_1)dt}\int\frac{d}{dt}\left(e^{tx+\int(A_0/A_1)dt}\right)dt.$$

* We remark that (12) reduces to $A_0(t)D(t)=0$, which has only the solution $D(t)\equiv 0$.

† The case $k=0$ is treated and solved in Pincherle (Bologna Paper), loc. cit., pp. 423–424.

‡ We apply Theorem 3.

We are assuming that (12) (for $k=1$) has no solution $D(t)$ which is analytic throughout $|t|<R$. Therefore, at some zero $t=\xi$ of $A_1(t)$, in $|t|<R$, $D(t)$ fails to be analytic. Let $t=\xi$ be a zero of order s, and set

$$\text{(a)} \qquad \frac{A_0(t)(t-\xi)^s}{A_1(t)} = \sum_{n=0}^{\infty} l_n(t-\xi)^n,$$

so that

$$\text{(b)} \qquad l_{s-1} = \frac{1}{(s-1)!}\frac{d^{s-1}}{dt^{s-1}}\left\{\frac{A_0(t)(t-\xi)^s}{A_1(t)}\right\}_{t=\xi};$$

$$\text{(c)} \qquad B(t) = \frac{A_0(t)(t-\xi)^s}{A_1(t)} - l_{s-1}(t-\xi)^{s-1},$$

$$\text{(d)} \qquad e^{\int B(t)(t-\xi)^{-s}dt} = \sum_{n=-\infty}^{\infty} s_n(t-\xi)^n,$$

$$\text{(e)} \qquad e^{(t-\xi)x+\int B(t)(t-\xi)^{-s}dt} = \sum_{n=-\infty}^{\infty} r_n(x)(t-\xi)^n,$$

so that

$$\text{(f)} \qquad r_n(x) = s_n + \frac{x}{1!}s_{n-1} + \frac{x^2}{2!}s_{n-2} + \cdots \qquad (n=0, \pm 1, \pm 2, \cdots).$$

Then

$$\text{(g)} \qquad e^{tx+\int(A_0/A_1)dt} = e^{\xi x}(t-\xi)^{l_{s-1}}\sum_{n=-\infty}^{\infty} r_n(x)(t-\xi)^n,$$

and

$$\text{(h)} \qquad \frac{d}{dt}\left(e^{tx+\int(A_0/A_1)dt}\right) = e^{\xi x}(t-\xi)^{l_{s-1}-1}\sum_{n=-\infty}^{\infty}(l_{s-1}+n)r_n(x)(t-\xi)^n.$$

Let us integrate (h), *introducing no "constant" of integration.* Two cases arise:

(α) l_{s-1} is not an integer (positive, negative, or zero). Then (h) integrates back into (g) and we have $A_0[y]+xA_1[y]=e^{tx}$. Therefore

$$\text{(i)} \qquad \Lambda(t;x) = e^{-\int[A_0(t)/A_1(t)]dt}\int\frac{e^{tx+\int[A_0(t)/A_1(t)]dt}}{A_1(t)}dt$$

is a resolving kernel. Here the integration is understood to be performed by expanding the integrand in a series about $t=\xi$, and introducting no "constant" of integration.

(β) l_{s-1} is an integer. Then the Laurent expansion (g) has a term

$e^{\xi x}r_{-l_s-1}(x)$ independent of t. Therefore on integrating (h) (and introducing no "constant" of integration),

$$\int \frac{d}{dt}(e^{tx+\int(A_0/A_1)dt})dt = e^{tx+\int(A_0/A_1)dt} - e^{\xi x}r_{-l_s-1}(x).$$

Let $u(x)$ be a solution* (of exp. val. $<R$) of

$$\text{(k)} \qquad A_0[u] + xA_1[u] = e^{\xi x}r_{-l_s-1}(x).$$

Then

$$\text{(l)} \qquad \Lambda(t\,;x) = e^{-\int[A_0(t)/A_1(t)]dt}\left[u(x) + \int\frac{e^{tx+\int[A_0(t)/A_1(t)]dt}}{A_1(t)}dt\right]$$

is a resolving kernel. Integration is as in (i).

To sum up:

THEOREM 9. *Let equation* (12) (*for* $k=1$) *have no solution* $D(t)$ *which is everywhere analytic in* $|t|<R$, *so that at some zero* $t=\xi$ *of* $A_1(t)$ (*in* $|t|<R$) $D(t)$ *is not analytic; and let the order of* $t=\xi$ *be s. Then a resolving kernel* $\Lambda\ (t;\,x)$ *for equation*

$$\text{(6b)} \qquad A_0[y] + xA_1[y] = f(x)$$

is given by (i) *or* (l), *according as* l_{s-1} *is not or is an integer* (*positive, negative, or zero*). *According to the case,* $f(x)$ (*of exp. val.* $<R$) *possesses the proper expansion*

$$\text{(26b}_1\text{)} \quad f(x) = \sum_{n=0}^{\infty}\left\{\frac{1}{2\pi i}\int_C\frac{F(t)}{t}\Theta_n\left(\frac{1}{t}\right)dt\right\}G_n(x),$$

$$\text{(26b}_2\text{)} \quad f(x) = \sum_{n=0}^{\infty}\left\{\frac{1}{2\pi i}\int_C\frac{F(t)}{t}\left[\Theta_n\left(\frac{1}{t}\right) + \Delta_n\left(\frac{1}{t}\right)dt\right\}G_n(x),$$

where

$$\Theta_n(t) = e^{-\int[A_0(t)/A_1(t)]dt}\int\frac{t^n e^{\int[A_0(t)/A_1(t)]dt}}{A_1(t)}dt,\quad \Delta_n(t) = u_n e^{-\int[A_0(t)/A_1(t)]dt},$$

and where u_n *is defined by* $u(x)=\sum_0^\infty u_n x^n/n!$. *And in either case a solution of* (6b) *is given by*

$$\text{(14b)} \qquad y(x) = \frac{1}{2\pi i}\int_C\frac{F(t)}{t}\Lambda\left(\frac{1}{t}\,;x\right)dt.$$

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* The right member of (k) is seen to be of exp. val. $<R$, so that by our hypothesis on $D(t)$, a solution $u(x)$ exists.

7. Partial differential equations of infinite order, constant coefficients. The equivalence between linear differential equations of infinite order and contour integral equations, such as has been established in the preceding sections, can be extended to other equations. In this section and the following one we consider two such extensions.

Denote by $A_{xy}[z(x, y)]$ the partial differential operator in two* independent variables

$$A_{xy}[z] \equiv \sum_{m,n=0}^{\infty} a_{mn} \frac{\partial^{m+n} z}{\partial x^m \partial y^n}, \tag{28}$$

where

$$A(t_1, t_2) = \sum_{m,n=0}^{\infty} a_{mn} t_1^m t_2^n.$$

Definition. If to the function $z(x; y)$, entire in x and y, there correspond two positive numbers λ_1, λ_2 such that for every $\epsilon > 0$,

$$\left| \frac{\partial^{m+n} z(0, 0)}{\partial x^m \partial y^n} \right| < M(\epsilon)(\lambda_1 + \epsilon)^m (\lambda_2 + \epsilon)^n,$$

then $z(x, y)$ will be said to be *of finite exponential value*; and if λ_1, λ_2 are the smallest numbers† for which this inequality holds, then $z(x, y)$ is *of exp. val.* (λ_1, λ_2).

Lemma 6. *If $A(t_1, t_2)$ is analytic in $|t_1| < R_1$, $|t_2| < R_2$, and $z(x, y)$ is of exp. val. (λ_1, λ_2) with $\lambda_1 < R_1$, $\lambda_2 < R_2$, then $A_{xy}[z]$ exists and is of exp. val. (μ_1, μ_2) where $\mu_1 \leqq \lambda_1$, $\mu_2 \leqq \lambda_2$.*

By hypothesis we have $|a_{mn}| < N(\epsilon)(R_1 - \epsilon)^{-m}(R_2 - \epsilon)^{-n}$, where

$$A(t_1, t_2) = \sum a_{mn} t_1^m t_2^n ;$$

and

$$z(x, y) \ll M(\epsilon) \sum (\lambda_1 + \epsilon)^m (\lambda_2 + \epsilon)^n x^m y^n/(m!n!) = M(\epsilon) e^{(\lambda_1+\epsilon)x + (\lambda_2+\epsilon)y}.$$

Therefore

$$A_{xy}[z] \ll M(\epsilon) N(\epsilon) e^{(\lambda_1+\epsilon)x+(\lambda_2+\epsilon)y} \sum_{m,n=0}^{\infty} \left(\frac{\lambda_1 + \epsilon}{R_1 - \epsilon}\right)^m \left(\frac{\lambda_2 + \epsilon}{R_2 - \epsilon}\right)^n,$$

and for ϵ sufficiently small the series on the right converges. The lemma now follows from the fact that e^{rx+sy} is of exp. val. $(|r|, |s|)$, and from the fact that $\epsilon > 0$ is arbitrary.

* The extension of the theory to k independent variables is immediate.

† If z is of finite exp. val. then two "smallest numbers" exist.

Lemma 7. *If* $A(t_1, t_2), B(t_1, t_2)$ *are analytic in* $|t_1|<R_1$, $|t_2|<R_2$, *and* $z(x, y)$ *is of exp. val.* $(\lambda_1, \lambda_2)<(R_1, R_2)$,* *then*

$$B_{xy}[A_{xy}[z]] = A_{xy}[B_{xy}[z]] = C_{xy}[z],$$

where $C(t_1, t_2)=A(t_1, t_2)B(t_1, t_2)$.

The identities are seen to be formally true, and the convergence of all the series involved can be established by the method of Lemma 6.

Lemma 8. *Let* $z(x, y)=\sum\lambda_{mn}x^m y^n/(m!n!)$ *be of exp. val.* $(\lambda_1, \lambda_2)<(R_1, R_2)$, *and set* $Z(x, y)=\sum\lambda_{mn}x^m y^n$. *Then* $Z(x, y)$ *is analytic in* $|x|<1/\lambda_1$, $|y|<1/\lambda_2$, *and*

$$z(x, y) = \left(\frac{1}{2\pi i}\right)^2 \int_{C_1}\int_{C_2} \frac{Z(t_1, t_2)}{t_1 t_2} e^{x/t_1+y/t_2}dt_1dt_2, \tag{29}$$

C_1, C_2 *being contours in the* t_1, t_2 *planes respectively, surrounding* $t_1=0$, $t_2=0$ *and lying in* $1/R_1<|t_1|<1/\lambda_1$, $1/R_2<|t_2|<1/\lambda_2$.

That $Z(x, y)$ is analytic in the region given follows at once. The integrand of (29) is then analytic on C_1, C_2, so that we may integrate, say first with respect to t_1, then t_2. Formula (29) results if we use relation (7) of §3.

If $(\lambda_1, \lambda_2)<(R_1, R_2)$ it is permissible to operate with A_{xy} under the integral sign, so that we have

Theorem 10. *If* $A(t_1, t_2)$ *is analytic in* $|t_1|<R_1$, $|t_2|<R_2$, *and* $f(x, y)$ *is of exp. val.* $(\lambda_1, \lambda_2)<(R_1, R_2)$, *then the partial differential equation*

$$A_{xy}[z] \doteq f(x, y) \tag{30}$$

is equivalent to the contour multiple integral equation†

$$f(x,y) = \left(\frac{1}{2\pi i}\right)^2 \int_{C_1}\int_{C_2} \frac{Z(t_1, t_2)}{t_1 t_2} A\left(\frac{1}{t_1}, \frac{1}{t_2}\right) e^{x/t_1+y/t_2}dt_1dt_2, \tag{31}$$

C_1, C_2 *being the contours of* (29).

The sense of the equivalence is as in Theorem 1: To every solution $z(x, y)=\sum\lambda_{mn}x^m y^n/(m!n!)$ of (30), of exp. val. $(m_1, m_2)<(R_1, R_2)$, corresponds a solution $Z(x, y)=\sum\lambda_{mn}x^m y^n$ of (31), analytic in $|t_1|<1/m_1$, $|t_2|<1/m_2$; and conversely. And each such solution is equivalent to a *proper expansion*‡

* The notation $(\lambda_1, \lambda_2)<(R_1, R_2)$ means that $\lambda_1<R_1$, $\lambda_2<R_2$.

† We use the identity $A_{xy}[e^{rx+sy}]=A(r, s)e^{rx+sy}$, $(|r|, |s|)<(R_1, R_2)$.

‡ For two variables a proper expansion is defined as one for which $\epsilon>0$ exists such that $|\lambda_{mn}|<M(\epsilon)(R_1-\epsilon)^m(R_2-\epsilon)^n$.

$$f(x, y) = \sum_{m,n=0}^{\infty} \lambda_{mn} G_{mn}(x, y) \tag{32}$$

where $\{G_{mn}(x, y)\}$ is the set of Appell polynomials in two variables defined by

$$e^{t_1 x + t_2 y} A(t_1, t_2) = \sum_{m,n=0}^{\infty} G_{mn}(x, y) t_1^m t_2^n. \tag{33}$$

As in the case of one independent variable, a solution of (30) can be obtained as a contour integral:

THEOREM 11.* *If $A(t_1, t_2)$ is analytic in $|t_1| < R_1$, $|t_2| < R_2$, and $f(x, y)$ is of exp. val. $(\lambda_1, \lambda_2) < (R_1, R_2)$, then a solution $z(x, y)$ of* (30), *of exp. val. $< (R_1, R_2)$, is given by*

$$z(x, y) = \left(\frac{1}{2\pi i}\right)^2 \int_{C_1} \int_{C_2} \frac{F(t_1, t_2)}{t_1 t_2} \cdot \frac{e^{x/t_1 + y/t_2}}{A\left(\frac{1}{t_1}, \frac{1}{t_2}\right)} dt_1 dt_2. \tag{34}$$

Here C_1, C_2 are the contours of (31) *and* (29), *and $F(t_1, t_2)$ is the function related to $f(t_1, t_2)$ in the same way that $Z(x, y)$ is related to $z(x, y)$.*

COROLLARY. *If $f(x, y)$ is of exp. val. $(\lambda_1, \lambda_2) < (R_1, R_2)$ it possesses the proper expansion*

$$f(x, y) = \sum_{m,n=0}^{\infty} \lambda_{mn} G_{mn}(x, y) \tag{35}$$

where

$$\lambda_{mn} = \left(\frac{1}{2\pi i}\right)^2 \int_{C_1} \int_{C_2} \frac{F(t_1, t_2)}{t_1^{m+1} t_2^{n+1}} \frac{dt_1 dt_2}{A(1/t_1, 1/t_2)}. \tag{36}$$

It is clear that the results of this section carry over at once to any number of independent variables.

8. **"Laurent" differential equations of infinite order, constant coefficients.** Let $A(t) \sim \sum_{-\infty}^{\infty} a_n t^n$ be a formal Laurent series, and denote by $A_L[y(x)]$ the "Laurent" differential operator

$$A_L[y] = \sum_{n=-\infty}^{\infty} a_n y^{(n)}(x),$$

where

$$y^{(-k)}(x) = \int_0^x \underset{(k\text{-fold})}{\cdots} \int_0^x y(x) dx \cdots dx, \quad k > 0.$$

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* The proof is immediate, using the relation (29) with f, F in place of z, Z.

Then we term

$$A_L[y] = f(x) \tag{37}$$

a "*Laurent*" *differential equation of infinite order.*

We make the following assumptions:

(a) $A(t) = \sum_{-\infty}^{\infty} a_n t^n$ converges, $0 < |t| < R$;

(b) $f(x)$ is of exponential value less than R.

LEMMA 9. *We have*

$$\int_0^x \cdots_{(n\text{-fold})} \int_0^x e^{tx} dx \cdots dx = \frac{1}{t^n} e^{tx} - \left\{ \frac{x^{n-1}}{(n-1)!} \frac{1}{t} + \frac{x^{n-2}}{(n-2)!} \frac{1}{t^2} + \cdots + \frac{x}{1!} \frac{1}{t^{n-1}} + \frac{1}{t^n} \right\}. \tag{38}$$

LEMMA 10. *For every t in $0 < |t| < R$,*

$$A_L[e^{tx}] = A(t)e^{tx} - \Theta(t\,;x) \tag{39}$$

where

$$\Theta(t\,;x) = \sum_{n=1}^{\infty} \frac{a_{-n}}{t^n} \left\{ \frac{(tx)^{n-1}}{(n-1)!} + \cdots + \frac{tx}{1!} + 1 \right\}; \tag{40}$$

and $\Theta(t; x)$ is analytic for every x and for every t in $|t| > 0$.

(39) is a consequence of (38). We need then only consider $\Theta(t; x)$. Suppose $0 < |t|$. Then $1 + tx/1! + \cdots + (tx)^{n-1}/(n-1)! \ll e^{|x|t}$, so that $\Theta(t; x) \ll e^{|x|t} \sum_1^{\infty} |a_{-n}|/t^n$, and since this last series converges for all $|t| > 0$, Θ is analytic for all $|t| > 0$ and for all x.

On using (7) of §3, and (39), we can express (37) as a *contour integral equation*:

$$f(x) = \frac{1}{2\pi i} \int_C \frac{Y(t)}{t} \left\{ A\left(\frac{1}{t}\right) e^{x/t} - \Theta\left(\frac{1}{t}\,; x\right) \right\} dt\,; \tag{41}$$

and (37), (41) are equivalent in the sense given by Theorem 1 for equations (6), (9).

Equation (41) can however be simplified. For from the definition of Θ we find that

LEMMA 11. *$\Theta(1/t; x)$ is analytic in t at $t = 0$, and vanishes there.*

We have

$$\Theta\left(\frac{1}{t}\,; x\right) = t \sum_{n=1}^{\infty} a_{-n} \left\{ \frac{x^{n-1}}{(n-1)!} + \frac{x^{n-2}}{(n-2)!} t + \cdots + \frac{x}{1!} t^{n-2} + t^{n-1} \right\}.$$

Now $|a_{-n}|\leqq M$, so that $\Theta(1/t;\ x)\ll Mt\{e^{|x|}+te^{|x|}+t^2e^{|x|}+\cdots\}=Mte^{|x|}\cdot(1-t)^{-1}$; and this proves the lemma.

COROLLARY. *$\Theta(1/t; x)$ is an entire function in both x and t.*

Since $\Theta(1/t; x)=0$ for $t=0$, it follows that

$$\frac{1}{2\pi i}\int_C \left\{\frac{Y(t)}{t}\right\}\Theta\left(\frac{1}{t}\,;\,x\right)dt = 0,$$

if $Y(t)$ is analytic in and on C. Consequently we have

THEOREM 12. *The "Laurent" equation (37) is equivalent (in the sense of Theorem 1) to the contour integral equation*

$$f(x) = \frac{1}{2\pi i}\int_C \frac{Y(t)}{t}A\left(\frac{1}{t}\right)e^{x/t}dt. \tag{42}$$

THEOREM 13. *If $y(x)=\sum_0^\infty \lambda_n x^n/n!$, of exp. val. $<R$, is a solution of (37), then $f(x)$ possesses the proper* Appell function expansion*

$$f(x) = \sum_0^\infty \lambda_n G_n(x)\ ; \tag{43}$$

and conversely.† *Here $G_n(x)$ is defined by*

$$e^{tx}A(t) = \sum_{n=-\infty}^{\infty} G_n(x)t^n. \tag{44}$$

* That is, lim sup $|\lambda_n|^{1/n}<R$.

† The proof is immediate.

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